Financial Maths – Annuities

An annuity is a ‘stream of payments’, each of equal value made at either the start or end of a period. If the payments are made at the end of a period, the annuity is said to be paid ‘in arrears’, while payments made at the start of a period are an ‘annuity due’.

For example: If you invest R100 per year over a period of three years, paid in arrears, how much will the investment be worth if the interest rate is 10% per annum?

The best way to consider this is to draw a timeline to show the payments over time:

In this example, we are trying to work out the future value of the investment, or the FV. Working backwards from the last payment, we can say that: \( FV = 100 + 100(1.1) + 100(1.1)^2 = 331 \)

The first 100 represents the final payment, the second the middle payment which has accrued 1 year of interest, and the final 100 represents the first payment which accrued two years of payment.

Now, cast your minds back to geometric progressions and their sum formulae. Look at the above equation; it is, in fact, a GP.

Therefore, we can generalise the formula for our future value to:

\[
FV = x \left( \frac{(1 + i)^n - 1}{i} \right)
\]

Where: 
- \( x \)=payment value
- \( n \)=number of periods
- \( i \)=interest rate

What if the same situation is taken, but this time the payments are made at the start of each period? Once again, draw a timeline:

This time, working from the first payment, we can determine that
\[ FV = 100(1.1) + 100(1.1)^2 + 100(1.1)^3 = 346.10 \]

However, we can simplify this by removing a common factor of (1.1), to give:
\[ FV = (1.1)(100 + 100(1.1) + 100(1.1)^2) \]

Which is merely the *in arrears* formula, only multiplied throughout by (1.1). Therefore, we can generalise it thus:
\[ FV = x(1 + i) \left( \frac{(1 + i)^n - 1}{i} \right) \]

These apply to finding the future value, where you pay a certain amount per period and receive a lump sum at the end. But what about a loan, where you first receive a lump sum, which is then paid off over time? For such a problem, we need to find the present value, or PV.

For these, we use the following formulae:

In arrears:  \[ PV = x \left( \frac{1-(1+i)^{-n}}{i} \right) \]

Annuity due:  \[ PV = x(1 + i) \left( \frac{1-(1+i)^{-n}}{i} \right) \]

When presented with a problem, if you do not know whether or not to use a present or future value formula, remember this:

- PV: lump sum first, then payments
- FV: payments first, then lump sum

Unless otherwise stated, problems will usually deal with an annuity paid in arrears.

On the next page are some examples for you to try. Good luck!
Annuities Examples:

1. You borrow R1 500 000 at an interest rate of 9% p.a. compounded monthly. You will repay R10 000 per month. How long will it take to repay the loan?

2. You want to buy a sports car for R1 400 000 and make monthly repayments for 4 years. The car loan is charged at 9% interest p.a. compounded monthly. How much will you pay per month?

3. You borrow R1 000 000 to spend at a casino. Interest is charged at 9% p.a. compounded monthly.
   a. How much will your monthly payments be if this is a 10 year loan?
   b. If you decide to settle after 2 years, how much will you have to pay?

Try these out before you look at the answers. If you get stuck and no amount of guessing will help you, then take a look.
Annuities Memo:

1. \[ PV = x \left( \frac{1 - (1+i)^{-n}}{i} \right) \]
   \[ 1500000 = 10000 \left( \frac{1 - (1+\frac{0.09}{12})^{-n}}{\frac{0.09}{12}} \right) \]
   \[ 150 = 1 - \left( 1 + \frac{0.09}{12} \right)^{-n} \]
   \[ \frac{9}{8} = 1 - \left( 1 + \frac{0.09}{12} \right)^{-n} \]
   \[ \frac{1}{8} = - \left( 1 + \frac{0.09}{12} \right)^{-n} \]

Here, you encounter a problem. In order to work out the value of \( n \), you need to use logs. However, it is not possible to take the log of a negative number. This means that you will never be able to pay off your debt, as it is accumulating interest in greater amounts than you are paying.

2. \[ PV = x \left( \frac{1 - (1+i)^{-n}}{i} \right) \]
   \[ 1400000 = x \left( \frac{1 - (1+\frac{0.09}{12})^{-48}}{\frac{0.09}{12}} \right) \]
   \[ 1400000 = x(40.18) \]
   \[ x = \frac{1400000}{40.18} \]
   \[ x = R34839.06 \]

3. a. \[ PV = x \left( \frac{1 - (1+i)^{-n}}{i} \right) \]
   \[ 1000000 = x \left( \frac{1 - (1+\frac{0.09}{12})^{-120}}{\frac{0.09}{12}} \right) \]
   \[ 1000000 = x(78.94) \]
   \[ x = R12667.58 \]

b. A quick way of working this out is to take the PV of the loan with 8 years left to go, as this would be the amount you would have to pay to settle your debt. 8 years = 96 months.

\[ PV = 12667.58 \left( \frac{1 - (1+\frac{0.09}{12})^{-96}}{\frac{0.09}{12}} \right) \]

\[ PV = R864669.23 \]